① Satake Isomorphism:

$F$ = local field, so either $F = \mathbb{F}_q((t))$ or $F/\mathbb{Q}_p$ finite.
uniformizer $\pi \in \mathcal{O}_F$, $\mathcal{O}_F/\pi = k$ finite field.

- $\underbrace{GL_n(F)}_{=G}$ is a locally compact topological gp. $\supseteq GL_n(\mathcal{O}_F) = K$.

~~Heck~~ Spherical Hecke alg: $\mathcal{H} := \mathbb{C}_c[K\backslash G/K]$

$= \{ f : G \to \mathbb{C}$ with compact support and bi-invariant under $K \}$.

$f(k_1 g k_2) = f(g) \quad \forall\, k_1, k_2 \in K$.

~~algebra~~ convolution: $(f_1 * f_2)(g) = \int_G f_1(h) \cdot f_2(h^{-1}g)\, dh$.

$dh$: bi-invariant Haar measure on $G$ making $dh(K) = 1$.

We'd like to understand $\mathcal{H}$.

- $\mathbb{C}$-basis: $G = \coprod_{\underline{l} \in \mathbb{Z}^n_+} K \cdot \pi^{\underline{l}} \cdot K$ (Cartan decomposition)

$\mathbb{Z}^n_+ = \{ \underline{l} \in \mathbb{Z}^n \mid l_1 \geq l_2 \geq \cdots \geq l_n \}$.

$\pi^{\underline{l}} = \begin{pmatrix} \pi^{l_1} & & & \\ & \pi^{l_2} & & \\ & & \ddots & \\ & & & \pi^{l_n} \end{pmatrix}$. $\quad \mathbb{1}_{K \cdot \pi^{\underline{l}} \cdot K} = f_{\underline{l}}$.

So $\mathcal{H} = \bigoplus_{\underline{l} \in \mathbb{Z}^n_+} \mathbb{C} \cdot \mathbb{1}_{K \pi^{\underline{l}} K}$

Let's try to understand the convolution.

$$f_{\underline{l}} * f_{\underline{m}} (\pi^{\underline{N}}) = \int_G f_{\underline{l}}(h) \cdot f_{\underline{m}}(h^{-1} \cdot \pi^{\underline{N}})\, dh.$$

$$h \in K\pi^{\underline{l}}K \qquad h^{-1} \cdot \pi^{\underline{N}} \in K\pi^{\underline{m}}K.$$

$$= \mu\left(K\pi^{\underline{l}}K \cap \pi^{\underline{N}} K \pi^{-\underline{m}} K\right).$$

Prop. $\mathcal{H}$ is commutative:

$$f_{\underline{l}} * f_{\underline{m}} (\pi^{\underline{N}}) = f_{\underline{m}} * f_{\underline{l}} (\pi^{\underline{N}})$$

taking transpose inverse preserves measure:

$$\mu(K\pi^{\underline{l}}K \cap \pi^{\underline{N}} K \pi^{-\underline{m}} K)$$

$$= \mu(K\pi^{-\underline{l}}K \cap \pi^{-\underline{N}} K \pi^{\underline{m}} K)$$

$$= \mu(\pi^{\underline{N}} K \pi^{-\underline{l}} K \cap K \pi^{\underline{m}} K).$$

Maybe one still wants to compute what the volume is. In order to do so, we need to understand this intersection:

Both $\pi^{\underline{N}} K \pi^{-\underline{m}} K$ & $K\pi^{\underline{l}}K$ are left $K$-coset.

Iwasawa decomposition: $G = B \times^{B(\mathcal{O})} K = \coprod_{\mathbb{Z}^n} N \times^{N(\mathcal{O})} K.$

here $B \subseteq GL_n$ is the subgp of upper triangular.

$N \subseteq B$ is the subgp of unipotent upper $\Delta$ ← diagonal 1.

abuse notation:

$B = B(F) \quad N = N(F).$

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"pf": any lattice can ~~be~~ have a basis of the form.

$$\Lambda \subseteq F^{\oplus n}$$

- $\lambda_1 = a_{11} \cdot e_1$

  $\lambda_2 = a_{12} \cdot e_1 + a_{22} \cdot e_2$

  $\vdots$

  $\lambda_n = \sum_{i=1}^{n} a_{1i} e_i$

- we may even assume $a_{ii} = \pi^{l_i}$.

So we need to understand which ~~elements~~ representatives

$$\left(\coprod_{\underline{l} \in \mathbb{Z}^n} N/N(\mathcal{O})\right) \cdot K \text{ can occur in}$$

$$K \pi^{\underline{m}} K.$$

Lemma: Those $\underline{l}$'s occur are exactly those $\underline{l}$ that

$$(*) \quad \underline{l}^{\sigma} \leq \underline{m} \text{ for some } \sigma \in S_n.$$

pf of necessity of $(*)$: assume $m_i \geq 0$.

in terms of lattices. $K\pi^{\underline{m}}K$ corresponds

$$\left\{ \text{lattice } \Lambda \subseteq \mathcal{O}^{\oplus n} = \Lambda_0, \text{ s.t. } \Lambda_0/\Lambda \cong \bigoplus_i \mathcal{O}/(\pi^{m_i}) \right\}$$

if $\Lambda$ has a basis of the form, then

or if $\Lambda \in \pi^{\underline{l}} \cdot N \cdot K$

$$0 \to \mathcal{O} \cdot e_1 / \mathcal{O} \cdot e_1 \cap \Lambda = \pi^{l_1} \cdot e_1 \longrightarrow \Lambda_0/\Lambda \longrightarrow \bar{\Lambda}_1 \to 0.$$

$$0 \to \mathcal{O} \cdot \bar{e}_2 / \mathcal{O} \cdot \bar{e}_2 \cap \bar{\Lambda}_1 \longrightarrow \bar{\Lambda}_1 \longrightarrow \Lambda_2 \to 0$$

$\vdots$

which means on $\mathcal{O}/(\pi^{m_i})$ we found an increasing filtration whose graded pieces are

$$gr^{\bullet}_{i} = \mathcal{O}/\pi^{l_i}$$

Exercise : show this $\Longrightarrow$ $\underline{l}^{\sigma} \leq \underline{m}$

and "=" holds iff the filtration splits.

e.g. $$0 \longrightarrow \mathcal{O}/(\pi^2) \longrightarrow \mathcal{O}/\pi \oplus \mathcal{O}/(\pi^3) \longrightarrow \mathcal{O}/(\pi^2) \longrightarrow 0.$$

$$1 \longmapsto (1, \pi)$$

$$(a, b) \longmapsto b - a\cdot\pi.$$

sufficiency also leaves as an exercise.

But let's try an example:

$$K\cdot\begin{pmatrix}\pi^3 & 0\\ 0 & 1\end{pmatrix}\cdot K = \begin{pmatrix}\pi^3 & \mathcal{O}/(\pi^3)\\ 0 & 1\end{pmatrix}\cdot K$$

$$\sqcup \begin{pmatrix}\pi^2 & (\mathcal{O}/(\pi^2))^*\\ & \pi\end{pmatrix}\cdot K$$

$$\sqcup \begin{pmatrix}\pi & (\mathcal{O}/(\pi))^*\\ & \pi^2\end{pmatrix}\cdot K \sqcup \begin{pmatrix}1 & 0\\ 0 & \pi^2\end{pmatrix}\cdot K.$$

(5) Satake transform:

given $f \in \mathcal{H}$. consider $S_f(t)$

$$= \delta(t)^{1/2} \int_N f(t \cdot n)\, dn \qquad dn(N(\mathcal{O})) = 1.$$

where $t \in T \subseteq G$ diagonal matrices.

$$\delta(t) \cdot dn = d(tnt^{-1})$$

or $\delta(t) = \left| \det(\mathrm{Ad}(t) \mid \mathrm{Lie}(N)) \right|$.

Prop. $S_f$ commutes with convolution, hence

$$\mathcal{H} \longrightarrow \left(\mathbb{C}_c\left[T(\mathcal{O})\backslash T / T(\mathcal{O})\right], *\right) \text{ homomorphism}$$

$$\overset{\shortparallel}{(\mathbb{C}_c[\mathbb{Z}^n], *)}.$$

pf: $\displaystyle \delta(t)^{1/2} \int_N \int_G f_1(h) \cdot f_2(h^{-1} t n)\, dh\, dn.$

$$\left( G/K = \coprod_{\underline{m} \in \mathbb{Z}^n} \pi^{\underline{m}} \cdot N/N(\mathcal{O}) \right)$$

$$= \delta(t)^{1/2} \int_N \sum_{\underline{m} \in \mathbb{Z}^n} \int_N f_1(\pi^{\underline{m}} \cdot \tilde{n}) \cdot f_2(\tilde{n}^{-1} \pi^{-\underline{m}} \cdot t \cdot n)\, d\tilde{n}\, dn.$$

change variable $\displaystyle = \delta(t)^{1/2} \sum_{\underline{m}} \int_N f_1(\pi^{\underline{m}} \cdot \tilde{n})\, d\tilde{n} \cdot \int_N f_2(\pi^{-\underline{m}} \cdot t \cdot n)\, dn.$

$$= (S_{f_1} * S_{f_2})(t).$$

Thm: Image of $\mathcal{S}$ lands in $S_n$-invariant functions.

and

$$\mathcal{S}: \mathcal{H} \xrightarrow{\cong} (\mathbb{C}_c[\mathbb{Z}^n], *)^{S_n}.$$

pf: as we've seen $\mathcal{S}_{f_{\underline{m}}}$ has support exactly those $\underline{\ell}$ with $\underline{\ell}^\sigma \leq \underline{m}$.

suffices to show $\mathcal{S}_f$ are $S_n$-invariant.

For $GL_2$:

$$K \cdot \begin{pmatrix} \pi^n & 0 \\ 0 & 1 \end{pmatrix} K = \coprod_{0 < m < n} \begin{pmatrix} \pi^m & (\mathcal{O}/(\pi^m))^* \\ 0 & \pi^{n-m} \end{pmatrix} \cdot K$$

$$\amalg \begin{pmatrix} \pi^n & (\mathcal{O}/\pi^n) \\ 0 & 1 \end{pmatrix} K \amalg \begin{pmatrix} 1 & 0 \\ 0 & \pi^n \end{pmatrix} \cdot K$$

take $\delta(t)^{1/2}$ into account one checks $S_2$-invariance directly.

For general $n$: ingenious change variable !!!

$$\delta(t)^{1/2} \cdot \int_N f(tn)\, dn \overset{n = t^{-1}\tilde{n}^{-1}t\tilde{n}}{=\!=\!=} \delta(t)^{1/2} \cdot \Psi(t) \cdot \int_N f(n^{-1}tn)\, dn.$$

assume $t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$ s.t. $\left|t_i/t_j - 1\right|_F \geq 1$.

$$\Psi(t) = \left| \det\left(1 - Ad(t)^{-1} \,\middle|\, Lie(N)\right)\right|$$

$\delta(t)^{1/2} \cdot \Psi(t)$ effect on $n_{ij} \in Lie(N)$.

$$\left|t_i/t_j\right|^{1/2} \cdot \max\left(1, \left|t_j/t_i\right|\right) = \left[\frac{\max(|t_i|, |t_j|)}{\min(|t_i|, |t_j|)}\right]^{1/2}$$

(7) Hence $\delta(t)^{1/2}\cdot\Psi(t)$ as a function on $T$ is $S_n$-inv!

$$\int_N f(n^{-1}tn)\,dn = \int_{T\backslash G} f(g^{-1}tg)\,dg.$$

~~change~~ $S_n = N(T)/T$, so $S_n$-invariance comes from left multiply $g$ by transition matrices.

Only caution: have to be careful of integration on $T\backslash G$, but anyway there's ton of theory which says one can do it.